The theory of elastic beams and its finite element implementation

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Summary
After the creation of the theory of elasticity by Cauchy, published in 1822, St.Venant developed the theory of tension, bending and torsion of beams based upon this theory of elasticity. The main results were published in 1855 and 1856.
Though the equations of the theory of elasticity were linearized, the results obtained by St.Venant for the stressdistribution and the stiffness properties remain valid for large displacements and large relative rotations as long as the local strains of a beam remain small. For materials used in structures, that may not be loaded beyond the elastic limit of the material, this condition is satisfied.
A brief account is given of the theory of the elastic line, applicable for the description of the mechanical behaviour of slender beams. This nonlinear theory has been applied to the stability analysis of rods and beams in its continuum mode. Thus a bifurcation point in the load-deformation relation of the geometrical perfect and physically homogeneous rods and beams under certain loadings could be established.
The column buckling, first studied by Euler (1744) before the creation of the mathematical theory of elasticity, was together with the general problem of elastic stability fully understood at a much later date (Koiter 1945).
Though the results, obtained with the continuum model of the elastic line, remained limited to the determination of the behaviour in the neighbourhood of the bifurcation point and to the analysis of the important effects of small imperfections, the finite element model now makes a realistic simulation of large displacements and rotations of elastic rods and beams a matter of straightforward desk computations. The type of finite element model, proposed by the author in 1981 [5], is eminently suitable for these computations and is therefore once again defined in this paper.

1. Stiffness properties and stressdistribution of prismatic beams.

The theory of St.Venant determines the stresses and deformations in a prismatic beam, made of an isotropic, linearly elastic material. The results are valid for beams of arbitrary length, but the loaddistributions at the ends of the beam follow from these results. The fact that actual loading of beams in structures will differ from the loaddistributions, dictated by St.Venant's solution of the problem, will however only affect the situation close to the ends of the beam, provided its cross-section is sufficiently solid. For beams with thin-walled, open cross-sections the disturbance due to non-conforming loaddistributions will propagate over a great length of the beam. But in all other cases the principle of St.Venant of elastic equivalence of statically equipollent systems of load is valid.

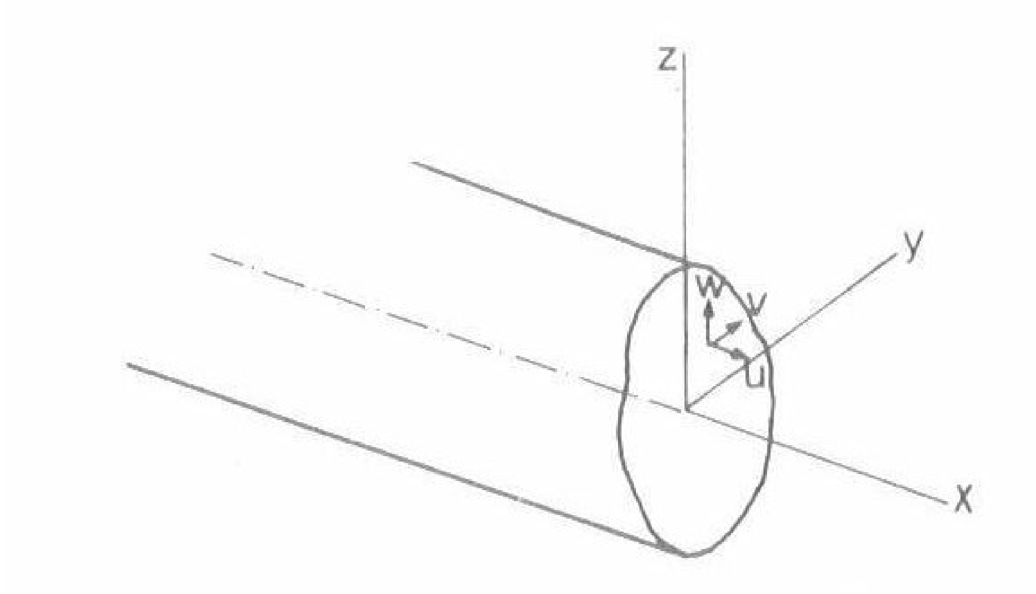

Fig.1 Prismatic beam, coordinate system and displacements.

In terms of the coordinate system shown in Fig.1, we shall give the results of St.Venant's solution. We have added the finite element solution of the warping displacements, that have to be determined for each shape of cross-section. In St.Venant's solution a warping displacement function of the cross-sectional coordinates has to satisfy a second order partial differential equation with an appropriate boundary condition.
The displacement components of a point of a cross-section at a coordinate x are given by the following expressions. The displacements, that leave the beam undeformed, are represented by the three translations, u_0, v_0, w_0, and by the three (small) rotations, $\varphi_0, \psi_0, \chi_0$. The two independent elasticity constants of the isotropic material are represented by the shear modulus G and Poisson's ratio ν. The warping displacement of the cross-section is given by u_w.

$$\begin{aligned}
u &= u_0 - \chi_0 y + \psi_0 z + \varepsilon_0 x + u_w + \tfrac{1}{2G(1+\nu)}\left[(\beta_0 x + \tfrac{1}{2}\beta x^2)y + (\gamma_0 x + \tfrac{1}{2}\gamma x^2)z\right],\\
v &= v_0 + \chi_0 x - (\varphi_0 + \omega x)z - \nu\varepsilon_0 y - \tfrac{\nu}{2G(1+\nu)}\left[\tfrac{1}{2}(\beta_0 + \beta x)(y^2 - z^2) + (\gamma_0 + \gamma x)yz\right]\\
&\quad - \tfrac{1}{2G(1+\nu)}(\tfrac{1}{2}\beta_0 x^2 + \tfrac{1}{6}\beta x^3), \qquad (1)\\
w &= w_0 - \psi_0 x + (\varphi_0 + \omega x)y - \nu\varepsilon_0 z - \tfrac{\nu}{2G(1+\nu)}\left[(\beta_0 + \beta x)yz - \tfrac{1}{2}(\gamma_0 + \gamma x)(y^2 - z^2)\right]\\
&\quad - \tfrac{1}{2G(1+\nu)}(\tfrac{1}{2}\gamma_0 x^2 + \tfrac{1}{6}\gamma x^3).
\end{aligned}$$

The expression for u_w consists of an unknown function of the section coordinates and an additional function of these coordinates in case Poisson's ratio ν is unequal to zero.

$$u_w = X(y,z) + \tfrac{\nu}{2(1+\nu)}(\tfrac{1}{6}\beta y^3 + \tfrac{1}{2}\beta yz^2 + \tfrac{1}{6}\gamma z^3 + \tfrac{1}{2}\gamma y^2 z) \qquad (2)$$

Since plane cross-sections of a beam generally do not remain plane in St.Venant's solution, obviously this solution can be rigidly valid only if the warping of the cross-

section is not hampered at the ends of the beam. In structures this will be seldom the case, but just for this reason the above mentioned principle of St.Venant is so important for practical applications of the stiffness properties of beams, derived from his solution of the beam deformation problem.
From the expressions for the displacement components we derive the results for the straincomponents and for the stresscomponents, determined by the straincomponents in accordance with the linear relations between stress and strain for an elastic, isotropic material.

$$
\begin{aligned}
&\varepsilon_x = \frac{\partial u}{\partial x} = \frac{\sigma_x}{2G(1+\nu)} = \varepsilon_0 + \frac{1}{2G(1+\nu)}\left[(\beta_0 + \beta x)y + (\gamma_0 + \gamma x)z\right],\\
&\varepsilon_y = \frac{\partial v}{\partial y}, \varepsilon_z = \frac{\partial w}{\partial z}, \varepsilon_y = \varepsilon_z = -\nu\varepsilon_0, \Rightarrow \sigma_y = \sigma_z = 0,\\
&\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\tau_{xy}}{G} = \frac{1}{G}\frac{\partial X}{\partial y} - \omega z + \frac{\nu}{2G(1+\nu)}\beta z^2, \qquad (3)\\
&\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \frac{\tau_{xz}}{G} = \frac{1}{G}\frac{\partial X}{\partial z} + \omega y + \frac{\nu}{2G(1+\nu)}\gamma y^2,\\
&\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = \frac{\tau_{yz}}{G} = 0.
\end{aligned}
$$

The values of the five parameters, $\varepsilon_0, \beta_0, \beta, \gamma_0, \gamma$, are determined by the stress resultants, the normal force N, the two components of the shearforce, D_y, D_z, and the two components of the bending moment, M_y, M_z. These relations are independent of the warping function *X*. The expression for the twisting moment, M_x, obviously does not depend on the warping function if the warping displacements are equal to zero, which is only the case for circularly symmetric cross-sections. Though the shear components D_y, D_z are the resultants of the shear stresses, which do depend on the function *X*, these resultants are independent of X because of the boundary condition:

$$\tau_{xy} n_y + \tau_{xz} n_z = 0. \qquad (4)$$

The other two boundary conditions,

$$\sigma_y n_y + \tau_{yz} n_z = 0,$$

$$\tau_{yz} n_y + \sigma_z n_z = 0,$$

are satisfied by the expressions (3). Note that along the length of the prismatic beam on the outer surface holds $n_x = 0$.
For axes through the geometrical centre of the cross-section ($\int_A y dA = 0, \int_A z dA = 0$) we define

$$I_y = \int_A z^2 dA, I_z = \int_A y^2 dA, C_{yz} = \int_A yz dA, I_p = I_y + I_z. \qquad (5)$$

We derive

$$N = \int_A \sigma_x dA = 2G(1+\nu)A\varepsilon_0,$$

$$M_y = \int_A \sigma_x z dA = (\beta_0 + \beta x)C_{yz} + (\gamma_0 + \gamma x)I_y, \tag{6}$$

$$M_z = -\int_A \sigma_x y dA = -(\beta_0 + \beta x)I_z - (\gamma_0 + \gamma x)C_{yz}.$$

The linearized expressions for the local curvatures of the beam axis are

$$\kappa_y = -\frac{\partial^2 w}{\partial x^2} = \gamma_0 + \gamma x,$$

$$\kappa_z = \frac{\partial^2 v}{\partial x^2} = \beta_0 + \beta x.$$

Thus in (6) we have the linear relation between bending moments and curvatures.
We consider

$$\int_A \frac{\partial \sigma_x}{\partial x} y dA = \beta I_z + \gamma C_{yz},$$

and note that by partial integration, making use of the equilibrium condition

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0, \tag{7}$$

we have

$$\int_A \frac{\partial \sigma_x}{\partial x} y dA = -\int_A \left(\frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z}\right) y dA = -\int_A \left\{\frac{\partial}{\partial y}(\tau_{xy} y) + \frac{\partial}{\partial z}(\tau_{xz} y) - \tau_{xy}\right\} dA =$$

$$-\int_{\partial A} (\tau_{xy} n_y + \tau_{xz} n_z) y ds + \int_A \tau_{xy} dA = \int_A \tau_{xy} dA = D_y.$$

Similarly the expression for D_z is derived. Hence we have, independent of the warping function *X*,

$$D_y = \beta I_z + \gamma C_{yz} = -\frac{dM_z}{dx},$$

$$D_z = \beta C_{yz} + \gamma I_y = \frac{dM_y}{dx}. \tag{8}$$

Substitution of the expressions for τ_{xy}, τ_{xz} from (3) into the equation of equilibrium (7) may lead us to the second order partial differential equation for the warping function *X*, where the boundary condition (4) supplies the appropriate boundary condition for this partial differential equation. However closedform solutions are for nearly all cross-sectional shapes out of the question.Therefore instead of showing this partial differential equation we shall describe the finite element solution of the warping problem. With a deskcomputer this solution has become a straightforward compution, that produces all relevant properties of a beamsection, together with nice illustrations of the stressdistributions and of the warping of the cross-section.
Our starting point will be the virtual power formulation of the remaining deformation problem of a beamsection.

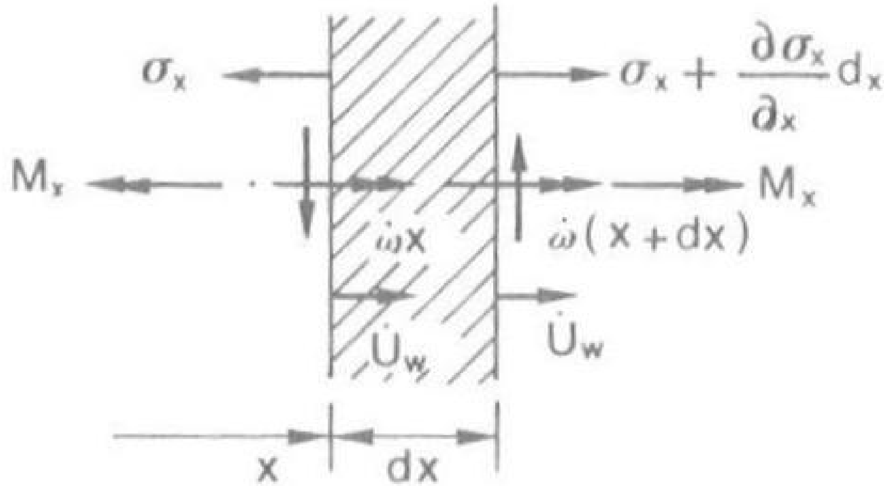

Fig.2 Slice of beam with stresses and velocities.

Since the warping function *X* and the specific angle of twist ω are independent of the longitudinal coordinate x it suffices to consider a slice dx of the beam, depicted in Fig.2. The principle of virtual power stipulates that the power of the stresses, acting on the two cross-sections, is equal to zero for all values of velocities that leave the slice of the beam undeformed. This is an equilibrium condition with subsidiary conditions, that can be taken into account by Lagrangian multipliers. We only have to consider arbitrary warping velocities $\dot{u}_w = \dot{X}/G$ and rates of twist $\dot{\omega}$. Only the shearstrains

$$\gamma_{xy} = \frac{1}{G}\frac{\partial X}{\partial y} - \omega z + \frac{\nu}{2G(1+\nu)}\beta z^2,$$

$$\gamma_{xz} = \frac{1}{G}\frac{\partial X}{\partial z} + \omega y + \frac{\nu}{2G(1+\nu)}\gamma y^2,$$

are affected by $\dot{X}, \dot{\omega}$. Taking the condition of zero deformation into account by the duals of $\dot{\gamma}_{xy}, \dot{\gamma}_{xz}$, equilibrium in accordance with the principle of virtual power requires:

$$dx\int_A \frac{\partial \sigma_x}{\partial x}\frac{\dot{X}}{G}dA + M_x\dot{\omega}dx = dx\int_A \left\{ \tau_{xy}\left(\frac{1}{G}\frac{\partial \dot{X}}{\partial y} - \dot{\omega}z\right) + \tau_{xz}\left(\frac{1}{G}\frac{\partial \dot{X}}{\partial z} + \dot{\omega}y\right) \right\} dA = 0 \forall \dot{X}, \dot{\omega}.$$

The multipliers τ_{xy}, τ_{xz} are ofcourse recognized as the shearstress components because for an elastic material the expression for the rate of work of deformation is equal to the rate of change of the elastic potential with a quadratic expression for the contribution of the shearstrains. This implies for the isotropic material:

$$\tau_{xy} = G\gamma_{xy}, \tau_{xz} = G\gamma_{xz}.$$

Substituting the expression for σ_x from (3) we arrive at the following variational formulation of the shear deformation problem for a prismatic beam:

$$\frac{1}{G}\int_A(\beta y+\gamma z)\dot{X}dA+M_x\dot{\omega}=\frac{1}{G}\int_A\left[\left(\frac{\partial X}{\partial y}-G\omega z+\frac{\nu}{2(1+\nu)}\beta z^2\right)\left(\frac{\partial \dot{X}}{\partial y}-G\dot{\omega}z\right)+\right.$$
$$\left.\left(\frac{\partial X}{\partial z}+G\omega y+\frac{\nu}{2(1+\nu)}\gamma y^2\right)\left(\frac{\partial \dot{X}}{\partial z}+G\dot{\omega}y\right)\right]dA\forall\dot{X},\dot{\omega}. \tag{9}$$

For the determination of the function X we first observe that it is a continuous function of the coordinates y,z, which permits a representation to any desired degree of accuracy by continuous, but only piecewise differentiable functions of these coordinates. Furthermore any shape of cross-section, simple or multiple connected, can be arbitrarily closely approximated by a composition of triangles. The finite element method consists in this case of dividing the cross-sectional area into a finite number of triangular elements with functions for X such that continuity at the element boundaries is ensured, while for each element these functions are unequivocally determined by independent parameters, expressing the function values at the nodal points. Though linear functions would be permissible, quadratic functions are preferable, because of the linearly distributed stresses they can describe. This is particularly important for thin walled sections with a linear distribution of shear stress over the thickness.

For triangular elements triangular coordinates are appropriate. With the rectangular coordinates y,z of the three cornerpoints of a triangle the cartesian coordinates are expressed in terms of the triangular coordinates as follows:

$$\begin{vmatrix}1\\y\\z\end{vmatrix}=\begin{bmatrix}1&1&1\\y_1&y_2&y_3\\z_1&z_2&z_3\end{bmatrix}\begin{vmatrix}L_1\\L_2\\L_3\end{vmatrix}$$

Since $L_1+L_2+L_3=1$ there are only two linearly independent triangular coordinates. The quadratic representation of the warping function for one element is now defined with the cornerpoint and midside values as parameters.

$$X=X^{eT}\begin{vmatrix}(2L_1-1)L_1\\(2L_2-1)L_2\\(2L_3-1)L_3\\4L_1L_2\\4L_2L_3\\4L_3L_1\end{vmatrix},X^e=\begin{vmatrix}X_1\\X_2\\X_3\\X_4\\X_5\\X_6\end{vmatrix}.$$

Differentiation and integration formulae can be given with the aid of the following coordinate differences:

$$a_1=y_3-y_2,b_1=z_2-z_3,$$
$$a_2=y_1-y_3,b_2=z_3-z_1,$$
$$a_3=y_2-y_1,b_3=z_1-z_2.$$

The finite element contributions to the equations resulting from the variational condition (9) can now be calculated by application of the differentiation and integration rules, given by:

$$\frac{\partial}{\partial y}=\sum_{i=1}^{3}\frac{\partial L_i}{\partial y}\frac{\partial}{\partial L_i}=\sum_{i=1}^{3}\frac{b_i}{2A}\frac{\partial}{\partial L_i},$$
$$\frac{\partial}{\partial z}=\sum_{i=1}^{3}\frac{\partial L_i}{\partial z}\frac{\partial}{\partial L_i}=\sum_{i=1}^{3}\frac{a_i}{2A}\frac{\partial}{\partial L_i},$$

with $2A=a_2b_1-a_1b_2=a_3b_2-a_2b_3=a_1b_3-a_3b_1$, while the integration rule reads

$$\int_A L_1^p L_2^q L_3^r dA=\frac{p!q!r!}{(p+q+r+2)!}2A.$$

When from hereon X denotes the vector of all nodal displacements in the whole cross-section, by calculating the contributions from the individual finite elements the variational condition (9) can be written in the form:

$$\dot{X}^T\left[\frac{b_z\beta+b_y\gamma}{G}\right]+\dot{\omega}M_x=\dot{X}^T\left[\frac{KX}{G}-b_w\omega+\frac{\nu}{2G(1+\nu)}(b_y^*\beta+b_z^*\gamma\right]+$$
$$\dot{\omega}[-X^Tb_w+GI_p\omega-\frac{\nu}{2(1+\nu)}(\beta I_{y3}-\gamma I_{z3})]\forall\dot{X},\dot{\omega}.$$

The matrix K, the vectors b_w,b_y,b_z,b_y^*,b_z^*, and the third order moments of the cross-section I_{y3},I_{z3}, are built from the contributions of the individual finite elements by the appropriate computer procedures. They define the linear equations

$$\begin{aligned}KX&=b_wG\omega+\left(b_z-\tfrac{\nu}{2(1+\nu)}b_y^*\right)\beta+\left(b_y-\tfrac{\nu}{2(1+\nu)}b_z^*\right)\gamma,\\ M_x&=-X^Tb_w+GI_p\omega-\tfrac{\nu}{2(1+\nu)}\left(\beta I_{y3}-\gamma I_{z3}\right).\end{aligned}\tag{10}$$

One of the nodal displacements in *X* must be prescribed, otherwise the warping displacements would be indeterminate and the matrix *K* will be singular.
The solution for *X* may now be written with separate terms containing $G\omega,\beta,\gamma.$

$$X=X_1G\omega+X_2\beta+X_3\gamma,$$

with

$$\begin{aligned}X_1&=K^{-1}b_w,\\ X_2&=K^{-1}\left(b_z-\tfrac{\nu}{2(1+\nu)}b_y^*\right),\\ X_3&=K^{-1}\left(b_y-\tfrac{\nu}{2(1+\nu)}b_z^*\right).\end{aligned}\tag{11}$$

Substituting these solutions into the expressions for M_x and τ_{xy},τ_{xz} we find

$$M_x=G\omega\left(I_p-X_1^Tb_w\right)+\beta\left(X_2^Tb_w-\tfrac{\nu}{2(1+\nu)}I_{y3}\right)+\gamma\left(X_3^Tb_w+\tfrac{\nu}{2(1+\nu)}I_{z3}\right),$$

and for the individual finite elements

$$\tau_{xy}=\begin{vmatrix}L_1 & L_2 & L_3\end{vmatrix}\left[D_1X^e-G\omega\begin{vmatrix}z_1\\z_2\\z_3\end{vmatrix}+\tfrac{\nu}{2(1+\nu)}\beta\begin{vmatrix}z_1\\z_2\\z_3\end{vmatrix}\begin{vmatrix}z_1 & z_2 & z_3\end{vmatrix}\begin{vmatrix}L_1\\L_2\\L_3\end{vmatrix}\right],$$
$$\tau_{xz}=\begin{vmatrix}L_1 & L_2 & L_3\end{vmatrix}\left[D_2X^e+G\omega\begin{vmatrix}y_1\\y_2\\y_3\end{vmatrix}+\tfrac{\nu}{2(1+\nu)}\gamma\begin{vmatrix}y_1\\y_2\\y_3\end{vmatrix}\begin{vmatrix}y_1 & y_2 & y_3\end{vmatrix}\begin{vmatrix}L_1\\L_2\\L_3\end{vmatrix}\right].$$

The matrices D_1,D_2 determine the corner values of the derivatives of the function *X* for the individual triangular elements.

$$D_1 = \frac{1}{2A}\begin{bmatrix} 3b_1 & -b_2 & -b_3 & 4b_2 & 0 & 4b_3 \\ -b_1 & 3b_2 & -b_3 & 4b_1 & 4b_3 & 0 \\ -b_1 & -b_2 & 3b_3 & 0 & 4b_2 & 4b_1 \end{bmatrix},$$

$$D_2 = \frac{1}{2A}\begin{bmatrix} 3a_1 & -a_2 & -a_3 & 4a_2 & 0 & 4a_3 \\ -a_1 & 3a_2 & -a_3 & 4a_1 & 4a_3 & 0 \\ -a_1 & -a_2 & 3a_3 & 0 & 4a_2 & 4a_1 \end{bmatrix}.$$

The specific angle of twist ω may now be expressed in terms of the twisting moment M_x and the values of β and γ, that are determined by the shear forces D_y and D_z according to (8). Note that the torsional stiffness constant of the cross-sectional area is given by

$$I_T = \frac{M_x}{G\omega}\left(D_y = D_z = 0\right) = I_p - X_1^T b_{w.} \tag{12}$$

For the determination of the coordinates of the shear centre (y_{sc}, z_{sc}) and of the values of the shearcoefficients k_y, k_{yz}, k_z the elastic energy of the sheardeformation is calculated and put equal to the quadratic expression for this energy in terms of the stress resultants, such that to each of these stress resultants corresponds a dual deformation quantity of the elastic line model of the beam ($I_T = Ai_T^2$).

$$\int_A \left[\frac{\tau_{xy}^2 + \tau_{xz}^2}{2G}\right] dA = \frac{1}{2GA}\begin{vmatrix} D_y & D_z & M_x \end{vmatrix} \begin{bmatrix} k_y + \frac{z_{sc}^2}{i_T^2} & k_{yz} - \frac{y_{sc} z_{sc}}{i_T^2} & \frac{z_{sc}}{i_T^2} \\ k_{yz} - \frac{y_{sc} z_{sc}}{i_T^2} & k_z + \frac{y_{sc}^2}{i_T^2} & \frac{y_{sc}}{i_T^2} \\ \frac{z_{sc}}{i_T^2} & \frac{y_{sc}}{i_T^2} & \frac{1}{i_T^2} \end{bmatrix} \begin{vmatrix} D_y \\ D_z \\ M_x \end{vmatrix}.$$

The six independent constants in the matrix have been chosen in a way that makes a clear phenomenological interpretation possible. The constant i_T^2 determines the torsional stiffness, y_{sc}, z_{sc} determine the point through which the shear forces must go in order to avoid twisting of the beams (the so-called shear centre), and the other three constants are the shearcoefficients, that determine the average shear angles $\overline{\gamma_y}, \overline{\gamma_z}$, produced by the shear forces D_y, D_z and by the axial moment about the geometrical centre of the cross-section M_x.

$$\begin{aligned} \overline{\gamma_y} &= \frac{1}{GA}\left[\left(k_y + \frac{z_{sc}^2}{i_T^2}\right) D_y + \left(k_{yz} - \frac{y_{sc} z_{sc}}{i_T^2}\right) D_z + \frac{z_{sc}}{i_T^2} M_x\right], \\ \overline{\gamma_z} &= \frac{1}{GA}\left[\left(k_{yz} - \frac{y_{sc} z_{sc}}{i_T^2}\right) D_y + \left(k_z + \frac{y_{sc}^2}{i_T^2}\right) D_z - \frac{y_{sc}}{i_T^2} M_x\right], \\ \omega &= \frac{1}{GAi_T^2}\left[z_{sc} D_y - y_{sc} D_z + M_x\right]. \end{aligned} \tag{13}$$

In terms of the solution for the warping displacements X_1 the coordinates of the shear centre are given by

$$y_{sc} = \frac{-C_{yz} X_1^T b_z + I_z X_1^T b_y}{I_y I_z - C_{yz}^2},$$
$$z_{sc} = \frac{-I_y X_1^T b_z + C_{yz} X_1^T b_y}{I_y I_z - C_{yz}^2}.$$

The expressions for the shear coefficients are rather complicated. Unlike the other quantities they have a slight dependence on Poisson's ratio. When the bending deformations are uncoupled by taking the so-called principal axes, for which $C_{yz} = 0$, as the coordinate axes, the shear deformations are in general not uncoupled. However for slender beams the shear deformations may be neglected compared to the bending deformations. We have

$$k_y = \frac{A}{\left(I_y I_z - C_{yz}^2\right)^2}\left(I_y^2 k_y^* - I_y C_{yz} k_{yz}^* + C_{yz}^2 k_z^*\right),$$
$$k_{yz} = \frac{A}{\left(I_y I_z - C_{yz}^2\right)^2}\left(-I_y C_{yz} k_y^* + \tfrac{1}{2}(I_y I_z - C_{yz}^2) k_{yz}^* - I_z C_{yz} k_z^*\right),$$
$$k_z = \frac{A}{\left(I_y I_z - C_{yz}^2\right)^2}\left(C_{yz}^2 k_y^* - I_z C_{yz} k_{yz}^* + I_z^2 k_z^*\right),$$

where

$$k_y^* = X_2^T\left(b_z + b_y^*\right) - \frac{1}{I_T}\left(X_1^T b_z X_1^T b_y^* + \tfrac{\nu}{2(1+\nu)} I_{y3}\right) + \left(\tfrac{\nu}{2(1+\nu)}\right)^2 I_{y4},$$
$$k_{yz}^* = X_2^T\left(b_y + b_z^*\right) + X_3^T\left(b_z + b_y^*\right) - \frac{1}{I_T}\left(X_1^T b_z X_1^T b_z^* + X_1^T b_y X_1^T b_y^* + \tfrac{\nu}{2(1+\nu)}(I_{y3} - I_{z3})\right),$$
$$k_z^* = X_3^T\left(b_z + b_y^*\right) - \frac{1}{I_T}\left(X_1^T b_y X_1^T b_z^* - \tfrac{\nu}{2(1+\nu)} I_{z3}\right) + \left(\tfrac{\nu}{2(1+\nu)}\right)^2 I_{z4}.$$

The solution procedure for St.Venant's problem of beam stresses and deformations, described above, has been implemented in the JAVA-program BEAMS. It was adopted from an earlier PASCAL version, that was used by the author in his lectures on strength and stiffness of structures, starting in 1986.

2.Elastic line model of rods and beams.

For sufficiently slender beams and rods the local strain components will remain small for even large curvatures of the beam axis and for large angles of twist. Then the stressdistribution in a cross-section will hardly differ from the stressdistribution according to St.Venant's solution of the linearized equations of the theory of elasticity. The beam or rod may be modelled as an elastic line with in each point stiffness properties derived from St.Venant's solution. The axial strain is determined by the normal force N, the local curvature of the elastic line by the bending moment, the specific angle of twist by the axial moment, and the average shear angles by the shear forces and the axial moment.

We shall consider a beam or rod that may be represented by an elastic line along the central axis (i.e. the line connecting the geometrical centres of the cross-sections). In the reference state we take the elastic line along the x-axis of a fixed cartesian system with the cross-section oriented such, that the y-axis and z-axis coincide with the principal axes of the prismatic beam or rod ($C_{yz} = 0$). With the unit base vectors $\mathbf{e}_x, \mathbf{e}_\mathbf{y}, \mathbf{e}_\mathbf{z}$ of the fixed cartesian system the position of an undeformed elastic line segment of length l is given by the radius vector $\mathbf{r}$,

$$\mathbf{r} = (x_0 + s)\mathbf{e}_x, 0 \le s \le l,$$

while $\mathbf{e}_y, \mathbf{e}_z$ determine the orientation of the principal axes of the cross-section.
In the deformed state the change of position of a point of the elastic line is determined by the displacement components u, v, w, while the rotations of the orthogonal triad with basevectors $\mathbf{e}_{x*}, \mathbf{e}_{y*}, \mathbf{e}_{z*}$ are described by means of angular coordinates ψ, ϑ, φ in three orthogonal transformations.

$$\begin{vmatrix}\mathbf{e}_{x*}\\ \mathbf{e}_{y*}\\ \mathbf{e}_{z*}\end{vmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & \cos\psi & \sin\psi\\ 0 & -\sin\psi & \cos\psi\end{bmatrix}\begin{bmatrix}\cos\vartheta & 0 & -\sin\vartheta\\ 0 & 1 & 0\\ \sin\vartheta & 0 & \cos\vartheta\end{bmatrix}\begin{bmatrix}\cos\varphi & \sin\varphi & 0\\ -\sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{bmatrix}\begin{vmatrix}\mathbf{e}_x\\ \mathbf{e}_y\\ \mathbf{e}_z\end{vmatrix}. \qquad (15)$$

The radius vector in the deformed state is given by

$$\mathbf{r} = (\mathrm{x}_0 + s + u)\mathbf{e}_x + v\mathbf{e}_y + w\mathbf{e}_z. \qquad (16)$$

Here s is the material coordinate on the elastic line and the angular coordinates ψ, ϑ, φ together with the displacement components u,v,w determine as functions of this material coordinate the geometry of the elastic line in the deformed state. We can distinguish six deformation components, $\varepsilon, \gamma_{y*}, \gamma_{z*}, \kappa_{x*}, \kappa_{y*}, \kappa_{z*}$, defined as follows:

$$\begin{aligned}
&2\varepsilon(ds)^2 = \mathbf{dr} \circ \mathbf{dr} - (\mathrm{ds})^2,\\
&\gamma_{y*}ds = \mathbf{e}_{y*} \circ \mathbf{dr},\\
&\gamma_{\mathbf{z}*}ds = \mathbf{e}_{z*} \circ \mathbf{dr},\\
&\kappa_{\mathrm{x}*} = -\mathbf{e}_{y*} \circ \frac{d\mathbf{e}_{z*}}{ds} = \mathbf{e}_{z*} \circ \frac{d\mathbf{e}_{y*}}{ds}, \qquad (17)\\
&\kappa_{y*} = -\mathbf{e}_{z*} \circ \frac{d\mathbf{e}_{x*}}{ds} = \mathbf{e}_{x*} \circ \frac{d\mathbf{e}_{z*}}{ds},\\
&\kappa_{z*} = -\mathbf{e}_{x*} \circ \frac{d\mathbf{e}_{y*}}{ds} = \mathbf{e}_{y*} \circ \frac{d\mathbf{e}_{x*}}{ds}.
\end{aligned}$$

Conditions under which large elastic curvatures and twist of a beam or rod may occur imply for most materials that the extension of the central axis as well as the shear angles γ_{y*}, γ_{z*} may be put equal to zero. We shall limit our discussion of the continuum model of the elastic line to this case. Consequently we have

$$\varepsilon = \tfrac{1}{2}\left(1+\frac{du}{ds}\right)^2 + \tfrac{1}{2}\left(\frac{dv}{ds}\right)^2 + \tfrac{1}{2}\left(\frac{dw}{ds}\right)^2 - \tfrac{1}{2} = 0,$$

$$\gamma_{y^*} = \left(-\cos\psi\sin\varphi + \sin\psi\sin\vartheta\cos\varphi\right)\left(1+\frac{du}{ds}\right) + \left(\cos\psi\cos\varphi + \sin\psi\sin\vartheta\sin\varphi\right)\frac{dv}{ds} +$$

$$+\sin\psi\cos\vartheta\frac{dw}{ds} = 0,$$

$$\gamma_{z^*} = \left(\sin\psi\sin\varphi + \cos\psi\sin\vartheta\cos\varphi\right)\left(1+\frac{du}{ds}\right) + \left(-\sin\psi\cos\varphi + \cos\psi\sin\vartheta\sin\varphi\right)\frac{dv}{ds} +$$

$$+\cos\psi\cos\vartheta\frac{dw}{ds} = 0.$$

From (15) and (17) we obtain the following expression for the specific twist

$$\kappa_{x^*} = \frac{d\psi}{ds} - \sin\vartheta\frac{d\varphi}{ds}. \qquad (18)$$

The expressions for the curvatures of the elastic line read according to (15) and (17)

$$\begin{aligned} \kappa_{y^*} &= \cos\psi\frac{d\vartheta}{ds} + \sin\psi\cos\vartheta\frac{d\varphi}{ds}, \\ \kappa_{z^*} &= -\sin\psi\frac{d\vartheta}{ds} + \cos\psi\cos\vartheta\frac{d\varphi}{ds}. \end{aligned} \qquad (19)$$

If we consider large curvatures about one of the principal axes of the cross-section, while the curvature and the rotation about the other principal axis remain small, then simplifications are possible.

For $|\varphi| << 1$ and $\left|\frac{dv}{ds}\right| << 1$ we derive by linearization with respect to φ and v:

$$\begin{aligned} &\left(1+\frac{du}{ds}\right) = \cos\vartheta, && \kappa_{x^*} = \frac{d\psi}{ds} - \sin\vartheta\frac{d\varphi}{ds}, \\ &\frac{dv}{ds} = \varphi\cos\vartheta, && \kappa_{y^*} = \cos\psi\frac{d\vartheta}{ds} + \sin\psi\cos\vartheta\frac{d\varphi}{ds}, \\ &\frac{dw}{ds} = -\sin\vartheta, && \kappa_{z^*} = -\sin\psi\frac{d\vartheta}{ds} + \cos\psi\cos\vartheta\frac{d\varphi}{ds}. \end{aligned} \qquad (20)$$

Similarly we obtain for $|\vartheta| << 1$ and $\left|\frac{dw}{ds}\right| << 1$ by linearization with respect to ϑ and w:

$$\begin{aligned} &\left(1+\frac{du}{ds}\right) = \cos\varphi, && \kappa_{x^*} = \frac{d\psi}{ds} - \vartheta\frac{d\varphi}{ds}, \\ &\frac{dv}{ds} = \sin\varphi, && \kappa_{y^*} = \cos\psi\frac{d\vartheta}{ds} + \sin\psi\frac{d\varphi}{ds}, \\ &\frac{dw}{ds} = -\vartheta, && \kappa_{z^*} = -\sin\psi\frac{d\vartheta}{ds} + \cos\psi\frac{d\varphi}{ds}. \end{aligned} \qquad (21)$$

For the determination of the bifurcation points of the initially straight rod and beams it is sufficient to consider expressions, in which only terms up to the second degree are retained.

$$\frac{du}{ds} = -\tfrac{1}{2}(\vartheta^2 + \varphi^2), \quad \kappa_{x*} = \frac{d\psi}{ds} - \vartheta \frac{d\varphi}{ds},$$
$$\frac{dv}{ds} = \varphi, \quad \kappa_{y*} = \frac{d\vartheta}{ds} + \psi \frac{d\varphi}{ds}, \tag{22}$$
$$\frac{dw}{ds} = -\vartheta, \quad \kappa_{z*} = \frac{d\varphi}{ds} - \psi \frac{d\vartheta}{ds}.$$

However in order to settle the question of stability at a bifurcation point terms up to the third degree will be needed as we shall see.

3. Bifurcation problems for rods and beams.

First we consider the wellknown problem problem of the buckling of a column in the x-z plane in a way that provides a basis for comparison with the finite element representation.

The end conditions for the Euler column are given by

$$\begin{aligned} &s = 0 \Rightarrow u = 0, w = 0, M_y = 0, \\ &s = l \Rightarrow N = -F, w = 0, M_y = 0. \end{aligned} \tag{23}$$

Because we restrict the theory of the elastic line here to the case, that the elastic potential of the beam is bending energy, the power of the external load F is stored as bending energy and we have

$$-F\dot{u}_l = \int_0^l M_y \frac{d\dot{\vartheta}}{ds} ds = \int_0^l EI_y \frac{d\vartheta}{ds}\frac{d\dot{\vartheta}}{ds} ds \forall \frac{d\dot{\vartheta}}{ds} \Rightarrow M_y = EI_y \frac{d\vartheta}{ds}. \tag{24}$$

The equilibrium condition for the beam according to the principle of virtual power requires zero power for all motion in the absence of deformation. For $\psi = 0$ this condition implies $\dot{u}_l = 0$, while according to (20) we have two other conditions for zero deformation. All conditions can be expressed in terms of $\dot{\vartheta}$:

$$\int_0^l \frac{d\dot{u}}{ds} ds = \dot{u}_l = -\int_0^l \sin\vartheta\dot{\vartheta} ds,$$
$$\frac{d\dot{\vartheta}}{ds} = 0,$$
$$\int_0^l \frac{d\dot{w}}{ds} ds = \dot{w}_l - \dot{w}_0 = 0 = -\int_0^l \cos\vartheta\dot{\vartheta} ds.$$

By the principle of virtual power we then have with M_y as a multiplier variable and Q as a multiplier constant :

$$F\int_0^l \sin\vartheta\dot{\vartheta} ds = \int_0^l M_y \frac{d\dot{\vartheta}}{ds} ds - Q\int_0^l \cos\vartheta\dot{\vartheta} ds \forall \dot{\vartheta}. \tag{25}$$

Substituting (24) , linearizing and by partial integration we obtain

$$\int_0^l -\left(EI_y \frac{d^2\vartheta}{ds^2} + F\vartheta + Q \right)\dot{\vartheta} ds = 0 \forall \dot{\vartheta}.$$

Here the end conditions (22) have been taken into account. From the variational condition we derive

$$EI_y \frac{d^2\vartheta}{ds^2} + F\vartheta - Q = 0. \tag{26}$$

The solution of the differential equation (26),

$$\vartheta = \hat{\vartheta}_1 \cos(\mu s) + \hat{\vartheta}_2 \sin(\mu s) + Q/F, \quad \mu^2 = \frac{F}{EI_y},$$

must satisfy the end conditions

$$s = 0 : M_y = EI_y \frac{d\vartheta}{ds} = 0, s = l : M_y = EI_y \frac{d\vartheta}{ds} = 0, \int_0^l \vartheta ds = 0.$$

The eigenvalue μ then follows from the requirement that the homogeneous equations

$$\begin{bmatrix} 0 & \mu & 0 \\ -\mu \sin \mu l & \mu \cos \mu l & 0 \\ \frac{1}{\mu} \sin \mu l & \frac{1}{\mu}(1 - \cos \mu l) & l \end{bmatrix} \begin{vmatrix} \hat{\vartheta}_1 \\ \hat{\vartheta}_2 \\ Q/F \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix}$$

must have a nontrivial solution. We derive

$$\mu^2 l \sin \mu l = 0 \Rightarrow \mu l = \pi, \qquad \hat{\vartheta}_2 = 0, \quad Q/F = 0.$$

Hence the bifurcation solution is given by

$$F_c = \mu^2 EI_y = \frac{\pi^2 EI_y}{l^2}, \qquad \vartheta_c = \hat{\vartheta}_1 \cos \mu s, \qquad u_c = 0, Q_c = 0. \tag{27}$$

In their linearized form the bifurcation equations do not give an indication about the stability at the bifurcation point. By keeping terms up to the third degree in the equilibrium condition we can determine the change of F with the amplitude of the bifurcation $\hat{\vartheta}$. We put

$$\begin{aligned} &F = F_c + F^r, \\ &\vartheta = \hat{\vartheta} \cos(\mu s), \\ &Q = 0. \end{aligned} \tag{28}$$

Linearizing with respect to the perturbations and taking into account the terms in $\hat{\vartheta}$ up to the third degree we obtain from (25) the variational equilibrium condition:

$$\int_0^l \left[E\, I_y \frac{d\vartheta}{ds} \frac{d\dot{\vartheta}}{ds} - \left(F_c + F^r\right)\left(\vartheta - \tfrac{1}{6}\vartheta^3\right)\dot{\vartheta} \right] ds = 0 \forall \dot{\vartheta}.$$

Substituting (28) we have

$$\left[\mu^2 EI_y \hat{\vartheta} \int_0^l \sin^2(\mu s) ds - \left(F_c + F^r\right)\left\{ \hat{\vartheta} \int_0^l \cos^2(\mu s) ds + \tfrac{1}{6} \hat{\vartheta}^3 \int_0^l \cos^4(\mu s) ds \right\} \right] \dot{\hat{\vartheta}} = 0 \forall \dot{\hat{\vartheta}}.$$

The result,

$$F^r = \tfrac{1}{8} F_c \hat{\vartheta}^2, \tag{29}$$

shows the stability of the Euler column at the bifurcation point, because

$$F = F_c + \tfrac{1}{8} F \hat{\vartheta}^2 >= F_c.$$

As a second example, for which the comparison between the continuum approach and the finite element model has been made, we shall discuss the lateral buckling of the end-loaded cantilever beam in the case that the curvature κ_{y*} may be put equal to zero.

$$
\begin{aligned}
&\kappa_{y*} = \cos\psi \frac{d\vartheta}{ds} + \sin\psi \frac{d\varphi}{ds} = 0,\\
&M_{z*} = EI_z \kappa_{z*} = EI_z \left(-\sin\psi \frac{d\vartheta}{ds} + \cos\psi \frac{d\varphi}{ds} \right),\\
&M_{x*} = GI_T \kappa_{x*} = GI_T \left(\frac{d\psi}{ds} - \vartheta \frac{d\varphi}{ds} \right),\\
&w_0 = -\int_0^l \frac{dw}{ds} ds = \int_0^l \vartheta ds.
\end{aligned}
\tag{30}
$$

By the principle of virtual power we have the following equilibrium condition, in which we have simplified the expression for κ_{z*} with the aid of the condition for κ_{y*}.

$$
\int_0^l \left[EI_z \frac{1}{\cos\psi} \frac{d\varphi}{ds} \frac{d}{dt} \left(\frac{1}{\cos\psi} \frac{d\varphi}{ds} \right) + GI_T \left(\frac{d\psi}{ds} - \vartheta \frac{d\varphi}{ds} \right) \frac{d}{dt} \left(\frac{d\psi}{ds} - \vartheta \frac{d\varphi}{ds} \right) \right] ds +
$$

$$
+ \int_0^l M \frac{d}{dt} \left(\cos\psi \frac{d\vartheta}{ds} + \sin\psi \frac{d\varphi}{ds} \right) ds + F \int_0^l \dot{\vartheta} ds = 0 \forall \dot{\psi}, \dot{\varphi}, \dot{\vartheta}.
$$

The multiplier M is of course the bending moment, against which the beam is considered to be infinitely stiff. Linearizing and by partial integration we obtain

$$
\begin{aligned}
&\int_0^l \left[\left(-EI_z \frac{d^2\varphi}{ds^2} - \psi \frac{dM}{ds} - M \frac{d\psi}{ds} \right) \dot{\varphi} + \left(-GI_T \frac{d^2\psi}{ds^2} + M \frac{d\varphi}{ds} \right) \dot{\psi} \right] ds +\\
&+ \int_0^l \left(-\frac{dM}{ds} + F \right) \dot{\vartheta} ds + \left(EI_z \frac{d\varphi}{ds} + M\psi \right) \dot{\varphi} \Big|_0^l + GI_T \frac{d\psi}{ds} \dot{\psi} \Big|_0^l + M \dot{\vartheta} \Big|_0^l = 0 \forall \dot{\psi}, \dot{\vartheta}, \dot{\varphi}.
\end{aligned}
\tag{31}
$$

Acoording to (30) and (31) we derive

$$
\begin{aligned}
&\frac{dM}{ds} = F, \quad s = 0 : M = 0 \Rightarrow M = Fs,\\
&EI_z \frac{d^2\varphi}{ds^2} + F\psi + Fs \frac{d\psi}{ds} = 0 \Rightarrow EI_z \frac{d\varphi}{ds} + Fs\psi = 0,\\
&GI_T \frac{d^2\psi}{ds^2} - Fs \frac{d\varphi}{ds} = 0 \Rightarrow \frac{d^2\psi}{ds^2} + \frac{F^2 s^2}{EI_z GI_T} \psi = 0,\\
&\Rightarrow \psi = \hat{\psi}_1 \left(1 - \frac{\lambda^2}{3.4} \xi^4 + \frac{\lambda^2}{3.4} \frac{\lambda^2}{7.8} \xi^8 - \frac{\lambda^2}{3.4} \frac{\lambda^2}{7.8} \frac{\lambda^2}{11.12} \xi^{12} + \ldots\ldots \right) +\\
&\qquad \hat{\psi}_2 \left(\xi - \frac{\lambda^2}{4.5} \xi^5 + \frac{\lambda^2}{4.5} \frac{\lambda^2}{8.9} \xi^9 - \frac{\lambda^2}{4.5} \frac{\lambda^2}{8.9} \frac{\lambda^2}{12.13} \xi^{13} + \ldots\ldots \right),\\
&\lambda^2 = \frac{F^2 l^4}{EI_z GI_T}, \quad \xi = \frac{s}{l}, \quad \alpha = \frac{GI_T}{EI_z},\\
&\xi = 0 : \frac{d\psi}{ds} = 0, \quad \xi = 1 : \psi = 0, \quad \Rightarrow \hat{\psi}_2 = 0, \lambda_c^2 = 16.10096,
\end{aligned}
$$

$$\xi = 0 : \frac{d\varphi}{d\xi} = 0, \quad \xi = 1 : \varphi = 0. \tag{32}$$

$$\Rightarrow \varphi = \alpha^{1/2} \lambda_c \hat{\psi}_1 \left[\tfrac{1}{2}\xi^2 \{1 - \frac{\lambda_c^2}{3.4} \tfrac{1}{3}\xi^4 + \frac{\lambda_c^2}{3.4}\frac{\lambda_c^2}{7.8} \tfrac{1}{5}\xi^8 - \frac{\lambda_c^2}{3.4}\frac{\lambda_c^2}{7.8}\frac{\lambda_c^2}{11.12} \tfrac{1}{7}\xi^{12} + ...\} - 0.3117614 \right]$$

Again in their linearized form the bifurcation equations do not give an indication about the stability at the bifurcation point. However with

$$\cos\psi \frac{d\vartheta}{ds} + \sin\psi \frac{d\varphi}{ds} = 0 \Rightarrow \vartheta = -\int_\xi^1 \frac{d\vartheta}{d\xi} d\xi = -\int_\xi^1 \left(-\psi \frac{d\varphi}{ds} - \tfrac{1}{3}\psi^3 \frac{d\varphi}{ds} \right) d\xi$$

the stability coefficient is readily obtained from (31) and (32) by numerical integration if the bifurcation mode is substituted and terms up to the third degree in $\hat{\psi}_1$ are retained. From

$$\int_0^1 \left(\frac{d\varphi}{d\xi} + \tfrac{1}{2}\psi^2 \frac{d\varphi}{d\xi} \right) \left(\frac{d\dot{\varphi}}{d\xi} + \psi \frac{d\varphi}{d\xi}\dot{\psi} + \tfrac{1}{2}\psi^2 \frac{d\dot{\varphi}}{d\xi} \right) d\xi +$$

$$+ \alpha \int_0^1 \left(\frac{d\psi}{d\xi} - \vartheta \frac{d\varphi}{d\xi} \right) \left(\frac{d\dot{\psi}}{d\xi} - \frac{d\varphi}{d\xi}\dot{\vartheta} - \vartheta \frac{d\dot{\varphi}}{d\xi} \right) d\xi + \frac{Fl^2}{\sqrt{EI_z GI_T}} \alpha^{1/2} \int_0^1 \dot{\vartheta} d\xi = 0 \forall \dot{\psi}, \dot{\vartheta}, \dot{\varphi},$$

we obtain

$$F = \lambda_c \sqrt{\frac{EI_z GI_T}{l^2}} \left[1 + (0.263\alpha + 0.635)\hat{\psi}_1^2 \right] - 0.423 F \hat{\psi}_1^2,$$

or, for

$$F = \lambda_c \sqrt{\frac{EI_z GI_T}{l^2}} + O(\hat{\psi}_1^2) = F_c + O(\hat{\psi}_1^2),$$

$$F = F_c \left[1 + (0.263\alpha + 0.212)\hat{\psi}_1^2 \right]. \tag{33}$$

This result matches the result given in [1].

4.Finite element model of rods and beams.

The interface of finite elements, based upon the elastic line concept, is a nodal point. The location of these nodal points and the orientation of an orthogonal triad, rigidly attached to each nodal point, can be given by the position vector with components $x_i = \xi_i + u_i$ and by the orthogonal transformation with components R_{i*j}, which transforms a vector in the x_{i-} system into a vector with respect to the local triad at the nodal point. For modified angular coordinates of Euler this transformation is defined in (15), but for the description of arbitrarily large rotations in finite element programs the four Euler parameters (which have to satisfy one condition, leaving three independent parameters) are to be preferred.

The six deformation parameters or generalized strains for an initially straight bar connection nodal points p and q were define in [2]. Their geometrical meaning is indicated in Fig.3.

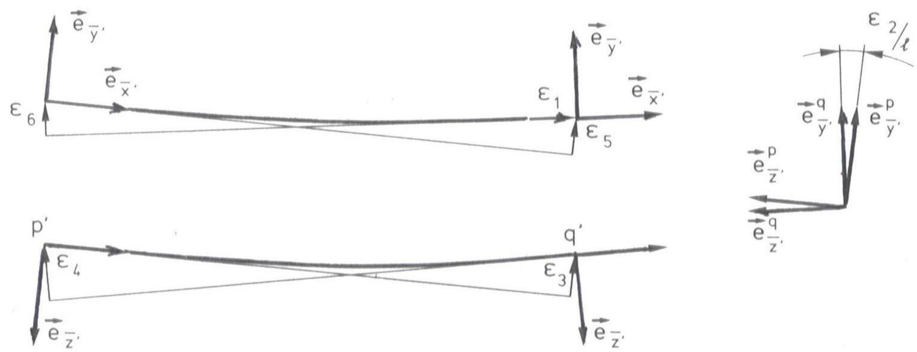

Fig.3 Generalized strains of bar element between nodal points p and q.

The six generalized strains can be expressed as analytic functions of the 2x6=12 nodal coordinates of the bar element e:

$$\varepsilon_i^e = D_i^e\left(u_k\right). \qquad (34)$$

Since the generalized strains of Fig.3 are defined with respect to orthogonal triads, oriented according to the bar-axis and the principal axes of its cross-section, we need the orthogonal matrix $R_{\bar{k}i} = R_{\bar{k}i*}$, which determines in the deformed and in the undeformed structure the orientation of the bar-axis and of the cross-sectional plane with reference to the local triads at the nodal points. The functions (34) for the generalized strains of Fig.3 can now be written as follows ($l^2 = \left(\xi_i^p - \xi_i^q\right)\left(\xi_i^p - \xi_i^q\right)$):

$$\begin{aligned}
\varepsilon_1 &= \frac{1}{2l}\left[\left(\xi_i^p - \xi_i^q + u_i^p - u_i^q\right)\left(\xi_i^p - \xi_i^q + u_i^p - u_i^q\right) - l^2\right] + \\
&\qquad \frac{1}{30l}\left(2\varepsilon_3^2 + \varepsilon_3\varepsilon_4 + 2\varepsilon_4^2 + 2\varepsilon_5^2 + \varepsilon_5\varepsilon_6 + 2\varepsilon_6^2\right), \\
\varepsilon_2 &= \tfrac{1}{2} l\left[R_{\bar{z}i*}R_{i*j}^p R_{jk*}^q R_{k*\bar{y}} - R_{\bar{y}i*}R_{i*j}^p R_{jk*}^q R_{k*\bar{z}}\right], \\
\varepsilon_3 &= -R_{\bar{z}i*}R_{i*j}^p\left(\xi_j^q - \xi_j^p + u_j^q - u_j^p\right), \\
\varepsilon_4 &= R_{\bar{z}i*}R_{i*j}^q\left(\xi_j^q - \xi_j^p + u_j^q - u_j^p\right), \\
\varepsilon_5 &= R_{\bar{y}i*}R_{i*j}^p\left(\xi_j^q - \xi_j^p + u_j^q - u_j^p\right), \\
\varepsilon_6 &= -R_{\bar{y}i*}R_{i*j}^q\left(\xi_j^q - \xi_j^p + u_j^q - u_j^p\right).
\end{aligned} \qquad (35)$$

The introduction of the quadratic terms in the bending deformations into the longitudinal deformation, first introduced in [4], make these expressions much more effective in a nonlinear analysis. In particular the number of finite elements needed to obtain a certain accuracy in a buckling analysis is drastically reduced by these terms, which give a contribution to the constant values of the second derivatives, decisive for the buckling phenomena.

If the deformations ε_i^e remain sufficiently small ($|\varepsilon_i^e| << l$), then in the elastic range they are linearly related to a normal force σ_1, a twisting moment σ_2 and four bending moments $\sigma_3, \sigma_4, \sigma_5, \sigma_6$ by a symmetric matrix of elasticity coefficents.

$$\sigma_i^e = S_{ij}^e\left(\varepsilon_i^e - \varepsilon_i^{e0}\right). \qquad (36)$$

Here we introduced initial deformations ε_i^{e0} to represent inelastic strains, or simply misfits in the perfect, undeformed structure ($u_k = 0$). For a prismatic bar with the *y*-

and z-axis in a cross-section along the principal axes the elasticity coefficients are given by the matrix

$$S^e = \begin{bmatrix} S_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & S_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4S_3 & -2S_3 & 0 & 0 \\ 0 & 0 & -2S_3 & 4S_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4S_4 & -2S_4 \\ 0 & 0 & 0 & 0 & -2S_4 & 4S_4 \end{bmatrix}, \tag{37}$$

with

$$S_1 = \frac{EA}{l}, \qquad S_2 = \frac{GI_T}{l^3},$$

$$S_3 = \frac{EI_{\bar{y}}}{l^3}, \qquad S_4 = \frac{EI_{\bar{z}}}{l^3}.$$

Here, like in par.2, the shear deformation has been neglected. If by means of a shear stiffness, a shear angle in the $\bar{x}-\bar{z}$ plane is taken into account by a shear deformation coefficient derived from St.Venant's solution of the stress problem, the submatrix with S_3 has to be replaced by

$$\frac{S_3}{1+12\beta}\begin{bmatrix} 4(1+3\beta) & -2(1-6\beta) \\ -2(1-6\beta) & 4(1+3\beta) \end{bmatrix}, \qquad \beta = \frac{k}{2(1+\nu)}\frac{I_{\bar{y}}}{l^2 A}. \tag{38}$$

As the expression for β shows, the contribution of the shear deformation can in general be neglected, because if $l^2 A >> I_{\bar{y}}$ does not hold the elastic line model of the rod or beam is questionable. It is not difficult however to take into account all elastic energy corresponding to St.Venant's solution of the stress problem. Then, in accordance with (13), the stiffness matrix S is the inverse of the flexibility matrix F, given below.

$$F = \begin{bmatrix} \frac{l}{EA} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{l^3}{GI_T} & \beta_{23} & -\beta_{23} & \beta_{25} & -\beta_{25} \\ 0 & \beta_{23} & \frac{l^3}{3EI_{\bar{y}}}+\beta_{33} & \frac{l^3}{6EI_{\bar{y}}}-\beta_{33} & -\beta_{35} & \beta_{35} \\ 0 & -\beta_{23} & \frac{l^3}{6EI_{\bar{y}}}-\beta_{33} & \frac{l^3}{3EI_{\bar{y}}}+\beta_{33} & \beta_{35} & -\beta_{35} \\ 0 & \beta_{25} & -\beta_{35} & \beta_{35} & \frac{l^3}{3EI_{\bar{z}}}+\beta_{55} & \frac{l^3}{6EI_{\bar{z}}}-\beta_{55} \\ 0 & -\beta_{25} & \beta_{35} & -\beta_{35} & \frac{l^3}{6EI_{\bar{z}}}-\beta_{55} & \frac{l^3}{3EI_{\bar{z}}}+\beta_{55} \end{bmatrix}$$

where

$$\beta_{23} = \frac{y_{sc} l^2}{GI_T}, \qquad \beta_{25} = \frac{z_{sc} l^2}{GI_T},$$

$$\beta_{33} = \frac{l}{GA}\left(k_2 + \frac{y_{sc}^2}{i_T^2}\right), \quad \beta_{35} = \frac{l}{GA}\left(k_{12} - \frac{y_{sc} z_{sc}}{i_T^2}\right), \quad \beta_{55} = \frac{l}{GA}\left(k_1 + \frac{z_{sc}^2}{i_T^2}\right).$$

It is clear from the corrective terms β that the coupling due to the excentricity of the shear centre (y_{sc}, z_{sc}) will have an influence rather than the other terms, if this excentricity is large. But this will be the case for open thin-walled cross-sections, for which the warping disturbance at the end points will again make the elastic line model questionable.
The generalized strains (34) possess the proper invariance with respect to rigid body movements of the bar element. However they are complicated functions of the nodal coordinates u_k. Since the value of the derivatives of these functions in the initial, undeformed state of the structure ($u_k = 0$) can easily be determined, in [2] the power series expansion was considered ($D_i^0 = 0$).

$$D_i = D_{i,k}^0 u_k + \tfrac{1}{2} D_{i,kl}^0 u_k u_l + \tfrac{1}{6} D_{i,klm}^0 u_k u_l u_m + \tfrac{1}{24} D_{i,klmn} u_k u_l u_m u_n + \ldots \quad . \tag{39}$$

In [2] the values of the derivatives $D_{i,k}^0$ up to $D_{i,klmn}^0$ are given.

The finite element model given here has been implemented in a JAVA-computer program SPFRAME for the static, kinematic and dynamic analysis of trusses and beam structures. This program was derived from an earlier PASCAL version (1987), that was used by the author for the instruction of students.

5.Finite element representation of bifurcation problems.

From the principle of virtual power we have for arbitrarily kinematically admissible $\dot{u}_k$

$$f_k \dot{u}_k = \sigma_i \dot{\varepsilon}_i = \sigma_i D_{i,k} \dot{u}_k \forall \dot{u}_k .$$

Hence the equations of equilibrium for the external forces on the structure read

$$f_k = \sigma_i D_{i,k} . \tag{40}$$

The multipliers σ_i are the generalized stresses, for an elastic material according to (36) determined by

$$\sigma_i = S_{ij}\left(\varepsilon_j - \varepsilon_j^0\right) = S_{ij}\left(D_i - \varepsilon_i^0\right). \tag{41}$$

Substitution of the series expansion (39) leads to equations with the following terms up to the fourth degree in u_k:

$$\begin{aligned} &D_{i,k}^0 S_{ij}\left[D_{j,l}^0 u_l + \tfrac{1}{2} D_{j,lm}^0 u_l u_m + \tfrac{1}{6} D_{j,lmn}^0 u_l u_m u_n + \tfrac{1}{24} D_{j,lmno}^0 u_l u_m u_n u_o\right] + \\ &-\left[D_{i,k}^0 + D_{i,kl}^0 u_l + \tfrac{1}{2} D_{i,klm}^0 u_l u_m + \tfrac{1}{6} D_{i,klmn}^0 u_l u_m u_n + \tfrac{1}{24} D_{i,klmno}^0 u_l u_m u_n u_o\right] S_{ij} \varepsilon_j^0 + \\ &+ \underline{D_{i,kl}^0 S_{ij} D_{j,m}^0 u_l u_m} + \tfrac{1}{2} D_{i,kl}^0 S_{ij} D_{j,mn}^0 u_l u_m u_n + \tfrac{1}{2} D_{i,klm}^0 S_{ij} D_{j,n}^0 u_l u_m u_n + \\ &+ \tfrac{1}{6} D_{i,klmn}^0 S_{ij} D_{j,o}^0 u_l u_m u_n u_o + \tfrac{1}{6} D_{i,kl}^0 S_{ij} D_{j,mno}^0 u_l u_m u_n u_o + \ldots\ldots = f_k . \end{aligned} \tag{42}$$

For a loading λf_k^1 we can solve the linearized equations

$$D_{i,k}^0 S_{ij} D_{j,l}^0 u_l^1 = f_k^1, \qquad D_{i,k}^0 S_{ij} D_{j,l}^0 u_l^0 = D_{i,k}^0 S_{ij} \varepsilon_j^0 . \tag{43}$$

Let

$$\sigma_i^1 = S_{ij} D_{j,k}^0 u_k^1, \qquad \sigma_i^0 = S_{ij}\left(D_{j,k}^0 u_k^0 - \varepsilon_j^0\right),$$

then σ_i^0 are the initial generalized stresses, u_k^0 are the geometrical imperfections of the structure, and σ_i^1 are the generalized stresses in equilibrium with the loads f_k^1 according to the linear theory.
In order that the solution of the linearized equations would satisfy the nonlinear equations rigorously, it would be required that

$$D_{i,kl}^0 u_l^1 = 0, D_{i,klm}^0 u_l^1 u_m^1 = 0, etc.,$$

and (44

$$D_{i,kl}^0 u_l^0 = 0, D_{i,klm}^0 u_l^0 u_m^0 = 0, etc.$$

Though in general this condition can only be approximately fulfilled for certain loads, the approximation can then be sufficiently close to consider the solution of the linearized equations as the fundamental state of deformation. However, this fundamental state of deformation need not be the only significant solution of the nonlinear equations.
If we substitute into the nonlinear equations a solution of the form

$$u_k = \lambda u_k^1 + u_k^0 + \Delta u_k,$$

then, as a consequence of (44) all terms linear in Δu_k disappear except the following

$$\left[D_{i,k}^0 S_{ij} D_{j,l}^0 + \left(\lambda \sigma_i^1 + \sigma_i^0 \right) D_{i,kl}^0 \right] \Delta u_l = 0. \tag{45}$$

But these equations may have nontrivial solutions ($\Delta u_k \neq 0$) for certain values of λ. The lowest value of λ for which the determinant of the matrix of coefficients in (45) is equal to zero (the lowest so-called eigenvalue) determines the buckling load of the structure.
The pure buckling phenomenon, characterized by a bifurcation of equilibrium states such as determined by eqs. (45), is a rather rare possibility. But in real structures under actual loading conditions and with geometrical imperfections the underlined terms in (42), which are responsible for the bifurcation phenomenon under the idealized conditions (44), play a predominant role in a perturbation analysis of the nonlinear equations.
An effective perturbation procedure is based upon the eigenvectors of the following eigenvalue problem:

$$K_{kl} e_l^p = -\lambda^p G_{kl} e_l^p, \text{ where } K_{kl} = D_{i,k}^0 S_{ij} D_{j,l}^0, G_{kl} = \sigma_i^1 D_{i,kl}^0. \tag{46}$$

Let there be r eigenvalues λ^p of finite value. We collect these eigenvalues and the associated eigenvectors in matrices with the following elements:

$$E_{kp} = e_k^p \text{ and } \Lambda_{pq} = \lambda^p \delta_{pq}, \quad p = 1, 2, 3, ..., r. \tag{47}$$

For normalized eigenvectors

$$E_{kp} G_{kl} E_{lq} = -\delta_{pq}, \quad E_{kp} K_{kl} E_{lq} = \Lambda_{pq}. \tag{48}$$

Now we express the nodal coordinate changes as follows:

$$u_k = \lambda u_k^1 + u_k^0 + E_{kp} \xi_p + u_k^r, \quad E_{kp} G_{kl} u_l^r = 0. \tag{49}$$

Equilibrium in terms of the principle of virtual power requires

$$\left[D_{i,k} S_{ij} \left(D_j - \varepsilon_j^0 \right) - \lambda f_k^1 \right] \left[\dot{\xi}_p \left(E_{kp} + u_{k,p}^r \right) + \dot{u}_k^r \right] = 0 \forall \dot{\xi}_p.$$

The $\dot{u}_k^r$ must, apart from being kinematically admissible, satisfy the orthogonality conditions, $\dot{u}_k^r G_{kl} E_{lp} = 0$.

The big gain of the perturbation analysis, considered here, is restricted to the case that the equilibrium equations stemming from $\dot{u}_k^r$ may be linearized with respect to u_k^r. Then with terms to the second degree in ξ_p we can solve u_k^* from

$$\begin{aligned} & K_{kl}u_l^* + \sigma_i^0 D_{i,kl}^0 \left(\lambda u_l^1 + u_l^0\right) + \sigma_i^0 D_{i,kl}^0 E_{lp}\xi_p + \\ & + \left[\tfrac{1}{2} D_{i,k}^0 S_{ij}\, D_{j,lm}^0 \left\{ \left(\lambda u_l^1 + u_l^0\right)\left(\lambda u_m^1 + u_m^0\right) + 2\left(\lambda u_l^1 + u_l^0\right) E_{mp}\xi_p + E_{lp}E_{mq}\xi_p\xi_q \right\} + \right. \\ & \left. + E_{lp} D_{i,kl}^0 S_{ij} D_{j,m}^0 E_{mq}\xi_p\xi_q + \tfrac{1}{2} E_{lp}E_{mq}\left(\lambda\sigma_i^1 + \sigma_i^0\right) D_{i,klm}^0 \xi_p\,\xi_q \right] + \ldots.. = 0 \end{aligned} \quad (50)$$

Since u_k^r must satisfy the orthogonality condition in (49) the proper solution for u_k^r is given by

$$u_k^r = \left(\delta_{kl} + E_{kp}E_{mp}G_{ml}\right) u_l^* . \quad (51)$$

We define

$$\begin{aligned} \sigma_i^r = S_{ij} D_{j,k}^0 u_k^r + \tfrac{1}{2} S_{ij} D_{j,kl}^0 \left\{ \left(\lambda u_k^1 + u_k^0\right)\left(\lambda u_l^1 + u_l^0\right) + 2\left(\lambda u_k^1 + u_k^0\right) E_{lp}\xi_p + \right. \\ \left. + E_{kp}E_{lq}\xi_p\,\xi_q \right\}. \end{aligned} \quad (52)$$

Now we obtain the nonlinear equations for ξ_p with λ as an independent loading parameter by considering the rates $\dot{\xi}_p$ in the principle of virtual power.

$$\begin{aligned} & E_{kp}\left(\lambda\sigma_i^1 + \sigma_i^0 + \sigma_i^r\right) D_{i,kl}^0 \left(\lambda u_l^1 + u_l^0\right) + \left(\Lambda_{pq} - \lambda\delta_{pq}\right)\xi_p + \\ & + E_{kp}E_{lq}\left(\sigma_i^0 + \sigma_i^r\right) D_{i,kl}^0 \xi_q + E_{kp}\sigma_i^0 D_{i,kl}^0 u_l^r + \\ & + u_{k,p}^r \left(\lambda\sigma_i^1 + \sigma_i^0\right) D_{i,kl}^0 u_l^r + E_{kp}\left(\lambda u_l^1 + u_l^0\right) D_{i,kl}^0 S_{ij} \left\{ \tfrac{1}{2} D_{j,mn}^0 E_{mq}E_{nr}\xi_q\xi_r + \right. \\ & \left. + \tfrac{1}{6} D_{j,mno}^0 E_{mq}E_{nr}E_{os}\xi_q\xi_r\,\xi_s \right\} + \left\{ E_{kp}\, D_{i,k}^0 S_{ij} D_{j,lm}^0 E_{mq} + E_{kp} D_{i,kl}^0 S_{ij} D_{j,m}^0 E_{mq} + \right. \\ & \left. + E_{kp}E_{mq}\left(\lambda\sigma_i^1 + \sigma_i^0\right) D_{i,klm}^0 \right\} u_l^r \xi_q + \left\{ \tfrac{1}{2} E_{kp} D_{i,k}^0 S_{ij} D_{j,lm}^0 E_{lq}E_{mr} + E_{kp}E_{lq} D_{i,kl}^0 S_{ij} D_{j,m}^0 E_{mr} + \right. \\ & \left. + \tfrac{1}{2} E_{kp}E_{lq}E_{mr}\left(\lambda\sigma_i^1 + \sigma_i^0\right) D_{i,klm}^0 \right\} \xi_q\xi_r + \left\{ \tfrac{1}{6} E_{kp} D_{i,k}^0 S_{ij} D_{j,lmn}^0 E_{lq}E_{mr}E_{ns} + \right. \\ & \left. + \tfrac{1}{2} E_{kp}E_{lq}E_{mr} D_{i,klm}^0 S_{ij} D_{j,n}^0 E_{ns} + \tfrac{1}{6} E_{kp}E_{lq}E_{mr}E_{ns}\left(\lambda\sigma_i^1 + \sigma_i^0\right) D_{i,klmn}^0 \right\} \xi_q\xi_r\xi_s = 0. \end{aligned} \quad (53)$$

Here only terms up to the third degree in ξ_p have been retained.

The participation factors ξ_p of the eigenvectors e_k^p will be relatively small as long as the load factor is far enough away from the corresponding eigenvalues λ^p. As a consequence in eqs.(53) only the participation factors for the eigenvalues close to λ need to be considered. Then in practical applications only a few nonlinear equations have to be solved in a nonlinear analysis, even for structures with a very large number of nodal coordinates.

Just as in the continuum approach of par.3 in the finite element representation of the Euler column only the first eigenvalue and eigenvector have to be taken into account for the determination of the stability coefficient. Since the linearized problem for the centrally loaded strut is so-called statically determinate the matrix of $D_{i,k}^0$ is a square, invertible matrix, and from (51),(52),(53) we obtain

$$\begin{aligned} & u_k^r = -\tfrac{1}{2}\left[D_{i,k}^0\right]^{-1} D_{i,lm}^0 e_l^1 e_m^1 \xi_1^2 - \left[D_{i,k}^0\right]^{-1}\left[S_{ij}^N\right]^{-1}\left[D_{j,l}^0\right]^{-1}\left\{ e_l^1\, D_{p,lm}^0 S_{pq}^B D_{q,n}^0 e_n^1 + \right. \\ & \left. + \tfrac{1}{2} e_m^1 e_n^1 \lambda\sigma_p^1\, D_{p,lmn}^0 \right\} \xi_1^2 , \end{aligned}$$

$$\sigma_i^r = -\left[D_{i,k}^0\right]^{-1} e_l^1 D_{p,kl}^0 S_{pq}^B D_{q,m}^0 e_m^1 \xi_1^2 - \tfrac{1}{2}\left[D_{i,k}^0\right]^{-1} e_l^1 e_m^1 \lambda \sigma_p^1 D_{p,klm}^0 \xi_1^2,$$

$$\begin{aligned}
&\left(\lambda^1 - \lambda\right)\xi_1 - 2e_k^1 e_l^1 D_{i,kl}^0 \left[D_{i,m}^0\right]^{-1} e_n^1 D_{p,mn}^0 S_{pq}^B D_{q,o}^0 e_o^1 \xi_1^3 + \\
&-e_k^1 e_l^1 D_{i,kl}^0 \left[D_{i,m}^0\right]^{-1} e_n^1 e_o^1 \lambda \sigma_p^1 D_{p,mno}^0 \xi_1^3 + \\
&+\tfrac{1}{2} e_k^1 e_l^1 D_{i,kl}^0 \left[D_{i,m}^0\right]^{-1} e_n^1 e_o^1 D_{j,no}^0 \left[D_{j,p}^0\right]^{-1} \lambda \sigma_q^1 D_{q,mp}^0 \xi_1^3 + \\
&+\tfrac{2}{3} e_k^1 D_{i,k}^0 S_{ij}^B D_{j,lmn}^0 e_l^1 e_m^1 e_n^1 \xi_1^3 + \tfrac{1}{6}\lambda \sigma_i^1 D_{i,klmn}^0 e_k^1 e_l^1 e_m^1 e_n^1 \xi_1^3 = 0.
\end{aligned} \qquad (54)$$

Here S_{ij}^N denotes that part of the stiffness matrix that is determined by S_1 and S_{ij}^B is the part defined by S_3.

In the expression for u_k^r the terms with $\left[S_{ij}^N\right]^{-1}$ are small as compared to the first term, but in the expression (52) for σ_i^r the contribution of this first term cancels, while even for $S_1 \to \infty$ the other terms give a finite contribution because of the multiplication by S_{ij}^N. In the last equation in (54) the terms with $\left[S_{ij}^N\right]^{-1}$, due to the substitution of u_k^r in (53), could be neglected.

In order to gain some insight in the expressions (54), and to make a comparison between the continuum approach and the finite element representation we shall introduce a notation, in which in the case of the Euler column the axial components u of the nodal displacements will be denoted by Latin indices, u_k, and the nodal deflections and rotations by Greek indices, u_α.

First we note that in the case of an inextensional axis we have instead of (49)

$$\begin{aligned}
u_k &= u_k^r, \\
u_\alpha &= E_{kp}\xi_p.
\end{aligned} \qquad (55)$$

The axial components of the displacements can now be expressed in terms of ξ_1 and e_α^1 with the aid of the expression for the axial deformation, $\varepsilon_i = D_i^N$.

$$\begin{aligned}
D_i^N = D_{i,k}^0 u_k^r + \tfrac{1}{2} D_{i,\alpha\beta}^0 e_\alpha^1 e_\beta^1 \xi_1^2 + \tfrac{1}{2} D_{i,kl}^0 u_k^r u_l^r + 3.\tfrac{1}{6} D_{i,k\alpha\beta}^0 u_k^r e_\alpha^1 e_\beta^1 + \\
\tfrac{1}{24} D_{\alpha\beta\gamma\delta}^0 e_\alpha^1 e_\beta^1 e_\gamma^1 e_\delta^1 \xi_1^4 + \ldots.. = 0.
\end{aligned}$$

Since $\left[D_{i,k}^0\right]$ in the case of the Euler column is a square non-singular matrix we find with terms up to the fourth degree in ξ_1:

$$\begin{aligned}
u_k^r = -\left[D_{i,k}^0\right]^{-1} \Big[\tfrac{1}{2} D_{i,\alpha\beta}^0 e_\alpha^1 e_\beta^1 \xi_1^2 - \tfrac{1}{4} D_{i,l\alpha\beta}^0 e_\alpha^1 e_\beta^1 \left[D_{j,l}^0\right]^{-1} D_{j,\gamma\delta}^0 e_\gamma^1 e_\delta^1 \xi_1^4 + \\
+ \tfrac{1}{24} D_{i,\alpha\beta\gamma\delta}^0 e_\alpha^1 e_\beta^1 e_\gamma^1 e_\delta^1 \xi_1^4 + \tfrac{1}{8} e_\alpha^1 e_\beta^1 D_{p,\alpha\beta}^0 \left[D_{p,l}^0\right]^{-1} e_\gamma^1 e_\delta^1 D_{q,\gamma\delta}^0 \left[D_{q,m}^0\right]^{-1} D_{i,lm}^0 \xi_1^4 \Big].
\end{aligned} \qquad (56)$$

If we substitute expression (56) into the equilibrium condition

$$\left(D_{i,k} u_{k,1}^r + D_{i,\alpha} e_\alpha^1\right) S_{ij}^B D_j - \lambda f_k^1 u_{k,1}^r = 0,$$

and if we observe that

$$\left[D_{i,k}^0\right]^{-1} f_k^1 = \sigma_i^1,$$

then again the last equation in (54) is obtained.

We can also, like in the continuum model of par.3, take the inextensibility condition, $D_i^N = 0$, into account as a subsidiary condition with the aid of a vector of multipliers

σ_i. We can express by the inextensibility condition the displacement rates $\dot{u}_k$ in the flexibility and rotation rates:

$$D^N_{i,k}\dot{u}_k + D^N_{i,\alpha}\dot{u}_\alpha = 0 \Rightarrow \dot{u}_k = -\left[D^N_{i,k}\right]^{-1} D^N_{i,\alpha}\dot{u}_\alpha .$$

Analogous to (25) we have according to the principle of virtual power

$$\left(-D_{i,k}\left[D^N_{p,k}\right]^{-1} D_{p,\alpha} + D_{i,\alpha}\right) S^B_{ij} D_j \dot{u}_\alpha +$$

$$+\sigma_i\left(-D^N_{i,k}\left[D^N_{p,k}\right]^{-1} D_{p,\alpha} + D^N_{i,\alpha}\right)\dot{u}_\alpha + \lambda f^1_k \left[D^N_{i,k}\right]^{-1} D_{i,\alpha}\dot{u}_\alpha = 0 \forall \dot{u}_\alpha .$$

The linearized bifurcation equations read

$$\begin{aligned} &\lambda\sigma^1_i D^0_{i,k} = \lambda f^1_k \Rightarrow \sigma^1_i = \left[D^0_{i,k}\right]^{-1} f^1_k , \\ &\left(D^0_{i,\alpha} S^B_{ij} D^0_{j,\beta} + \lambda^1\sigma^1_i D^0_{i,\alpha\beta}\right) e^1_\beta = 0, \qquad (57) \\ &D^0_{i,k} u_k = 0 \Rightarrow u_k = 0. \end{aligned}$$

Next we consider the bifurcations

$$\begin{aligned} &u_\alpha = e^1_\alpha \xi_1 + O\left(\xi^3_1\right), \\ &u_k = u^r_k , \\ &\sigma_i = \lambda\sigma^1_i + \sigma^r_i . \end{aligned}$$

We derive with terms up to the second degree in ξ_1 :

$$\begin{aligned} &u^r_k = -\tfrac{1}{2}\left[D^0_{i,k}\right]^{-1} D^0_{i,\alpha\beta} e^1_\alpha e^1_\beta \xi^2_1 , \\ &\sigma^r_i = -\left[D^0_{i,k}\right]^{-1}\left[\tfrac{1}{2}\lambda\sigma^1_p D^0_{p,k\alpha\beta} e^1_\alpha e^1_\beta \xi^2_1 + e^1_\alpha D^0_{p,k\alpha} S^B_{pq} D^0_{q,\beta} e^1_\beta \xi^2_1\right]. \end{aligned}$$

Finally it is required that the virtual power equation, including the subsidiary condition, is satisfied for arbitrary $\dot{\xi}_1$. The equation with terms up to the third degree in ξ_1, that follows from the condition

$$\left[\left(e^1_\alpha D^0_{i,\alpha} + e^1_\alpha D^0_{i,k\alpha} u^r_k + \tfrac{1}{2} D^0_{i,\alpha\beta\gamma} e^1_\alpha e^1_\beta e^1_\gamma \xi^3_1\right) S^B_{ij}\left(D^0_{j,\beta} e^1_\beta \xi_1 + 2.\tfrac{1}{2} D^0_{j,l\beta} u^r_l e^1_\beta \xi_1 + \right.\right.$$

$$\left.\tfrac{1}{6} D^0_{j,\alpha\beta\gamma} e^1_\alpha e^1_\beta e^1_\gamma \xi^3_1\right) + \left(\lambda\sigma^1_i + \sigma^r_i\right)\left(D^0_{i,k} u^r_{k,1} + D^0_{i,\alpha\beta} e^1_\alpha e^1_\beta \xi_1 + D^0_{i,kl} u^r_k u^r_{l,1} + D^0_{i,k\alpha\beta} u^r_k e^1_\alpha e^1_\beta \xi_1 + \right.$$

$$\left.\left.\tfrac{1}{2} D^0_{i,k\alpha\beta} u^r_{k,1} e^1_\alpha e^1_\beta \xi^2_1\right) - \lambda f^1_k u^r_{k,1}\right]\dot{\xi}_1 = 0 \forall \dot{\xi}_1 ,$$

is again identical to the last equation in (54). We recall that this equation was derived for a large, but finite stiffness against elongation of the column axis.

In the analysis of the continuum elastic line model as well as in the finite element analysis the stability problem could be solved even for the case of an inextensible column axis, because of the fact that the normal force and its first perturbation is statically determinate. A similar situation arises in the lateral buckling case of the end-loaded cantilever beam. Here in addition to the inextensibility condition a condition of zero curvature about a principal axis of the cross-section is taken into account, either directly or with the aid of the bending moment about this axis as lagrangian multiplier. This bending moment is also statically determinate in the example of lateral buckling considered in par.3.

The distribution of primary stresses (*N* in the case of the Euler column, *N* and *M* in the case of the end-loaded cantilever beam) can in other cases be statically indeterminate. Then there is only a well posed stability problem if the stiffness against these stresses is finite and the stability coefficient will depend largely on the redistribution of the primary stresses during buckling. This redistribution of stress can be determined by a model with a number of finite elements that is small as compared to the number of elements required to take the kinematical restraints into account with sufficient accuracy in the case of the Euler column and of the end-loaded cantilever beam. As it has been shown in [3] in these cases 16 elements were needed for a result with a 2% accuracy. The explanation for the fact, that in the case of statical indeterminacy a much smaller number of elements is needed to obtain this kind of accuracy, is given by the observation that all terms of the third degree, appearing in the last equation in (54), can usually be neglected because much larger terms of the third degree appear. These terms no longer cancel against each other in the case of statical indeterminacy.

Concluding remarks.

The classical subject of the theory of elastic beams and rods has received a practically important addition in the formulation of a finite element representation. Now nonlinear problems can be solved by straightforward desk computations.

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